ON QUANTUM GROUPS AND THEIR POTENTIAL USE IN MATHEMATICAL CHEMISTRY

Maurice KIBLER

*Institut de Physique Nucléaire de Lyon, IN2P3-CNRS et Université Claude Bernard,
43 Boulevard du 11 Novembre 1918, F-69622 Villeurbanne Cedex, France*

and

Tidjani NÉGADI

*Laboratoire de Physique Théorique, Institut des Sciences Exactes,
Université d'Oran Es-Sénia, 31100 Es-Sénia, Algeria*

Abstract

The quantum algebra $su_q(2)$ is introduced as a deformation of the ordinary Lie algebra $su(2)$. This is achieved in a simple way by making use of q-bosons. In connection with the quantum algebra $su_q(2)$, we discuss the q-analogues of the harmonic oscillator and the angular momentum. We also introduce q-analogues of the hydrogen atom by means of a q-deformation of the Pauli equations and of the so-called Kustaanheimo–Stiefel transformation.

1. Introduction

A new algebraic structure, the structure of quantum group, has been developed since 1985 [1–3] and is still the subject of developments both in mathematics and theoretical physics. Such a structure, which is related to the structure of Hopf bialgebra, takes its origin in various fields of theoretical physics (e.g., statistical mechanics, integrable systems, conformal field theory).

The notion of quantum group is more easily approached through the one of quantum algebra. Loosely speaking, the latter notion corresponds to a deformation, depending on a certain parameter q, of a Lie algebra. Most of the applications of quantum algebras, of potential use for chemical physics, have been mainly devoted to the harmonic oscillator [4–8] and to coherent states [9,10].

It is the aim of this paper to briefly describe one of the simplest quantum groups, viz., the quantum group $SU_q(2)$, or rather its quantum algebra $su_q(2)$, and to underline its potential use in chemical physics. For this purpose, we examine in turn three dynamical systems connected with quantum groups: the q-deformed harmonic oscillator, the q-deformed angular momentum and the q-deformed hydrogen atom. These new systems, also referred to as q-analogues, reduce to the corresponding ordinary systems in the limiting case $q = 1$.

The paper presents a review character as far as the q-analogues of the harmonic oscillator (in section 2) and the angular momenta (in section 3) are concerned. The discussion (in section 3 and appendix 2) about the relevance of the quantum algebra $so_q(3, 2)$ for studying the q-analogues of spherical and hyperbolic angular momenta is new. The introduction (in section 4) of q-analogues for the hydrogen atom is developed for the first time. No sophisticated mathematical pre-requisite is necessary to understand this self-contained article.

2.   *q*-analogue of the harmonic oscillator

We start with the usual Fock space

$$\mathcal{F} = \{|n\rangle : n \in \mathbb{N}\}, \tag{1}$$

which is very familiar to the chemist.

DEFINITION 1

Let us define the linear operators a^+, a and N on the vector space $\mathcal{F}$ by the relations

$$a^+|n\rangle = \sqrt{[n+1]}\,|n+1\rangle, \quad a|n\rangle = \sqrt{[n]}\,|n-1\rangle, \quad N|n\rangle = n|n\rangle \tag{2}$$

with $a|0\rangle = 0$, where we use the notation

$$[c] \equiv [c]_q = \frac{q^c - q^{-c}}{q - q^{-1}} = \frac{\sinh(c\ln q)}{\sinh(\ln q)}, \quad c \in \mathbb{C} \tag{3}$$

for a given q in the field of complex numbers $\mathbb{C}$.

It is to be observed that in the limiting case $q = 1$, we have simply $[c] = c$, so that a^+, a and N are, respectively, in this case the ordinary creation, annihilation and number operators encountered in various areas of theoretical chemistry and physics. In the case where $q \neq 1$, with q not being a root of unity, the operators a^+, a and N defined by eqs. (2),(3) are called q-deformed creation, annihilation and number operators, respectively. (In this case, the complex number $[c]$ defined by (3) is a q-deformed number; some algebraic relations satisfied by such q-deformed numbers are listed in appendix 1.)

Property 1

As a trivial property, we have

$$(a)^\dagger = a^+, \quad (N)^\dagger = N, \quad [N, a^+] = a^+, \quad [N, a] = -a, \tag{4}$$

where $(X)^\dagger$ denotes the adjoint of the operator X and $[X, Y] \equiv [X, Y]_- = XY - YX$ the commutator of X and Y.

Property 2

As a basic property, we can check that

$$aa^+ = [N + 1], \quad a^+a = [N], \quad aa^+ - q^{-1}a^+a = q^N, \quad aa^+ - qa^+a = q^{-N}, \tag{5}$$

where we use the abbreviation

$$[X] \equiv [X]_q = \frac{q^X - q^{-X}}{q - q^{-1}} = \frac{\sinh(X \ln q)}{\sinh(\ln q)}, \quad X \in \mathcal{F}, \tag{6}$$

which parallels for operators the defining relation (3) for numbers.

The set $\{a, a^+\}$ satisfying (4)–(6) is a set of q-bosons as originally defined by Mcfarlane [4] and Biedenharn [5] (see also refs. [6,7]). From eq. (5), it is clear that the operators a and a^+ reduce to ordinary bosons in the limiting case $q = 1$.

We are now in a position to introduce a q-deformed harmonic oscillator. The literature on this subject is now abundant and the reader may consult, for example, refs. [4–10] for further details.

DEFINITION 2

From the q-deformed creation and annihilation operators a and a^+, let us define the operators

$$p_x = i\sqrt{\hbar\mu\omega/2}\,(a^+ - a), \quad x = \sqrt{\hbar/2\mu\omega}\,(a^+ + a) \tag{7}$$

acting on $\mathcal{F}$, where $\hbar$, μ and ω have their usual meaning in the context of the (ordinary) harmonic oscillator.

Equation (7) defines q-deformed momentum and position operators p_x and x, respectively, and bears the same form as for the ordinary creation and annihilation operators corresponding to the limiting case $q = 1$.

Property 3

The commutator of the q-deformed operators x and p_x is

$$[x, p_x] = i\hbar([N + 1] - [N]), \tag{8}$$

which reduces to the ordinary value $i\hbar$ in the limiting case $q = 1$.

In terms of eigenvalues, eq. (8) can be rewritten as

$$[x, p_x] = i\hbar \frac{\cosh[(n + \frac{1}{2})\ln q]}{\cosh(\frac{1}{2}\ln q)} \tag{9}$$

when $q \neq 1$. Thus, we may think of a q-deformed uncertainty principle: the right-hand side of (9) increases with n (i.e. with the energy, see eq. (11) below) and is minimum as well as n-independent in the limiting case $q = 1$ [5].

DEFINITION 3

We define the self-adjoint operator H on $\mathcal{F}$ by

$$H = \tfrac{1}{2}\mu^{-1}p_x^2 + \tfrac{1}{2}\mu\omega^2 x^2 = \tfrac{1}{2}(a^+a + aa^+)\hbar\omega = \tfrac{1}{2}([N] + [N+1])\hbar\omega \qquad (10)$$

in terms of the q-deformed operators previously defined.

In the limiting case $q = 1$, the operator H is nothing but the Hamiltonian for a one-dimensional harmonic oscillator. Following Macfarlane [4] and Biedenharn [5], we take eq. (10) as the defining relation for a q-deformed one-dimensional harmonic oscillator. The case of a q-deformed d-dimensional, with $d \geq 2$, (isotropic or anisotropic) harmonic oscillator can be handled from a superposition of one-dimensional q-deformed oscillators.

Property 4

The spectrum of H is given by

$$E \equiv E_n = \tfrac{1}{2}([n] + [n+1])\hbar\omega = [2]_{q^{1/2}} \tfrac{1}{2}[n + \tfrac{1}{2}]\hbar\omega, \qquad n \in \mathbb{N} \qquad (11)$$

and is discrete.

This spectrum turns out to be a deformation of the one for the ordinary one-dimensional harmonic oscillator corresponding to the limiting case $q = 1$. The levels are shifted (except the ground level) when we pass from $q = 1$ to $q \neq 1$: the levels are not uniformly spaced.

3. q-analogues of angular momenta

We now continue with the Hilbert space

$$\mathcal{E} = \{\,|jm\rangle : 2j \in \mathbb{N}, m = -j(1)j\,\} \qquad (12)$$

spanned by the common eigenvectors of the z-component and the square of a generalized angular momentum.

DEFINITION 4

We define the operators a_+, a_+^+, a_- and a_-^+ on the vector space $\mathcal{E}$ by the relations

$$a_+ \, | \, jm \rangle = \sqrt{[j+m]} \, | j - \tfrac{1}{2}, m - \tfrac{1}{2} \rangle,$$

$$a_+^+ \, | \, jm \rangle = \sqrt{[j+m+1]} \, | j + \tfrac{1}{2}, m + \tfrac{1}{2} \rangle,$$

$$a_- \, | \, jm \rangle = \sqrt{[j-m]} \, | j - \tfrac{1}{2}, m + \tfrac{1}{2} \rangle,$$

$$a_-^+ \, | \, jm \rangle = \sqrt{[j-m+1]} \, | j + \tfrac{1}{2}, m - \tfrac{1}{2} \rangle, \tag{13}$$

where the numbers of the type $[c]$ are given by (3).

In the limiting case $q = 1$, eq. (13) gives back the defining relations used by Schwinger [11] in his (Jordan–Schwinger) approach to angular momentum (see also ref. [12]). By introducing

$$n_1 = j + m, \quad n_2 = j - m, \quad n_1 \in \mathbb{N}, \quad n_2 \in \mathbb{N} \tag{14}$$

and

$$| \, jm \rangle \equiv | j + m, j - m \rangle = | n_1 n_2 \rangle \in \mathcal{F}_1 \otimes \mathcal{F}_2, \tag{15}$$

eq. (13) can be rewritten in the form

$$a_+ \, | n_1 n_2 \rangle = \sqrt{[n_1]} \, | n_1 - 1, n_2 \rangle,$$

$$a_+^+ \, | n_1 n_2 \rangle = \sqrt{[n_1 + 1]} \, | n_1 + 1, n_2 \rangle,$$

$$a_- \, | n_1 n_2 \rangle = \sqrt{[n_2]} \, | n_1, n_2 - 1 \rangle,$$

$$a_-^+ \, | n_1 n_2 \rangle = \sqrt{[n_2 + 1]} \, | n_1, n_2 + 1 \rangle. \tag{16}$$

Therefore, the sets $\{a_+, a_+^+\}$ and $\{a_-, a_-^+\}$ are two commuting sets of q-bosons. More precisely, we can prove that

$$a_+ a_+^+ - q^{-1} a_+^+ a_+ = q^{N_1}, \quad a_- a_-^+ - q^{-1} a_-^+ a_- = q^{N_2},$$

$$[a_+, a_-] = [a_+^+, a_-^+] = [a_+, a_-^+] = [a_+^+, a_-] = 0, \tag{17}$$

with

$$N_1 | n_1 n_2 \rangle = n_1 | n_1 n_2 \rangle, \quad N_2 | n_1 n_2 \rangle = n_2 | n_1 n_2 \rangle \tag{18}$$

defining the number operators N_1 and N_2.

DEFINITION 5

Let us consider the operators

$$J_- = a_-^+ a_+, \quad J_3 = \tfrac{1}{2}(N_1 - N_2), \quad J_+ = a_+^+ a_- \tag{19}$$

defined in terms of q-bosons.

Property 5

The action of the linear operators J_-, J_3 and J_+ on the space $\mathcal{E}$ is described by

$$J_- |jm\rangle = \sqrt{[j+m][j-m+1]} \, |j, m-1\rangle,$$

$$J_3 |jm\rangle = m |jm\rangle,$$

$$J_+ |jm\rangle = \sqrt{[j-m][j+m+1]} \, |j, m+1\rangle, \tag{20}$$

a result that follows from (13) and (19).

The operators J_- and J_+ are clearly shift operators for the quantum number m. The operators J_-, $J_3 = (J_3)^\dagger$ and $J_+ = (J_-)^\dagger$ reduce to ordinary spherical angular momentum operators in the limiting case $q = 1$. The latter assertion is evident from (20) or even directly from (19).

At this stage, the quantum algebra $su_q(2)$ can be introduced, in a pedestrian way, from eqs. (19) and (20) as a deformation of the ordinary Lie algebra of the special unitary group $SU(2)$. In this regard, we have the following property.

Property 6

The commutators of the q-deformed spherical angular momentum operators J_-, J_3 and J_+ are

$$[J_3, J_-] = -J_-, \quad [J_3, J_+] = +J_+, \quad [J_+, J_-] = [2J_3], \tag{21}$$

which reduce to the familiar expressions known in angular momentum theory in the limiting case $q = 1$.

Equation (21) is at the root of the definition of the quantum algebra $su_q(2)$. Roughly speaking, this algebra is spanned by any set J_-, J_3, J_+ of three operators satisfying (21) where we recognize familiar commutators except for the third one. The notion of invariant operator also exists for quantum algebras. In this connection, we can verify that the operator

$$J^2 = \tfrac{1}{2}(J_+ J_- + J_- J_+) + \frac{[2]}{2} [J_3]^2 \tag{22}$$

is a Casimir operator in the sense that it commutes with each of the generators J_-, J_3 and J_+ of the quantum algebra $su_q(2)$. It can be proved that the eigenvalues of the Hermitian operator J^2 are $[j][j+1]$ with $2j \in \mathbb{N}$, a result compatible with the well-known one corresponding to the limiting case $q = 1$.

DEFINITION 6

We now introduce the operators

$$K_- = a_+ a_-, \quad K_3 = \tfrac{1}{2}(N_1 + N_2 + 1), \quad K_+ = a_+^+ a_-^+, \tag{23}$$

which are indeed q-deformed hyperbolic angular momentum operators.

Property 7

The action of the operators K_-, K_3 and K_+ on the space $\mathcal{E}$ is described by

$$K_- |jm\rangle = \sqrt{[j-m][j+m]}\ |j-1, m\rangle,$$

$$K_3 |jm\rangle = (j + \tfrac{1}{2})|jm\rangle,$$

$$K_+ |jm\rangle = \sqrt{[j-m+1][j+m+1]}\ |j+1, m\rangle, \tag{24}$$

a result to be compared to (20).

The operators K_- and K_+ behave like shift operators for the quantum number j. The operators K_-, $K_3 = (K_3)^\dagger$ and $K_+ = (K_-)^\dagger$ reduce to ordinary hyperbolic angular momentum operators in the limiting case $q = 1$ [11, 12]. From eq. (24), we expect that they generate the quantum algebra $su_q(1, 1)$, a result which is trivial when $q = 1$.

Property 8

The commutators of the q-deformed hyperbolic angular momentum operators K_-, K_3 and K_+ are

$$[K_3, K_-] = -K_-, \quad [K_3, K_+] = +K_+, \quad [K_+, K_-] = -[2K_3], \tag{25}$$

which characterizes the quantum algebra $su_q(1, 1)$.

Equations (20) and (21), on one hand, and eqs. (24) and (25), on the other, can serve to develop the theory of q-deformed spherical and hyperbolic angular momenta. This theory involves the q-deformation of coupling (Clebsch–Gordan) coefficients and recoupling (Racah and Wigner) coefficients, projection operators, etc., and shall not be described here (see, among numerous papers, ref. [13]). In the

limiting case $q = 1$, the Wigner–Racah algebra of $SU(2)$, in an $SU(2) \supset U(1)$ basis, plays a considerable rôle in this theory; in this case, the Lie algebra of the de Sitter group $SO(3, 2)$ is the natural framework for studying the Wigner–Racah algebra of $SU(2)$. We devote the rest of this section to some basic elements indicating the relevance of the quantum algebra $so_q(3, 2)$ when $q \neq 1$.

DEFINITION 7

We define the operators

$$k_+^+ = -a_+^+ a_+^+, \quad k_-^+ = a_-^+ a_-^+, \quad k_-^- = -a_+ a_+, \quad k_+^- = a_- a_-$$ (26)

in terms of q-bosons.

Property 9

The action of the operators k_+^+, k_-^+, k_-^- and k_+^- on the space $\mathcal{E}$ is described by

$$k_+^+ |jm\rangle = -\sqrt{[j + m + 1][j + m + 2]} \,|j + 1, m + 1\rangle,$$

$$k_-^+ |jm\rangle = \sqrt{[j - m + 1][j - m + 2]} \,|j + 1, m - 1\rangle,$$

$$k_-^- |jm\rangle = -\sqrt{[j + m - 1][j + m]} \,|j - 1, m - 1\rangle,$$

$$k_+^- |jm\rangle = \sqrt{[j - m - 1][j - m]} \,|j - 1, m + 1\rangle,$$ (27)

so that they act like mixed step operators for the quantum numbers j and m.

Some further properties, of interest for the quantum algebra $so_q(3, 2)$, of the operators of type J, K and k are relegated to appendix 2.

4. q-analogue of the hydrogen atom

We now consider an ordinary hydrogen-like atom in three dimensions with reduced mass μ and nuclear charge Ze. Here we deal only with the discrete spectrum of this (Coulomb) dynamical system, i.e., with negative energies E.

According to Pauli [14], the Coulomb system can be described in an operator form by the equations (see also ref. [15])

$$A^2 - B^2 = 0, \quad E(2A^2 + 2B^2 + \hbar^2) = -\tfrac{1}{2}\mu Z^2 e^4.$$ (28)

In eq. (28), the operators $A^2 = \sum_i A_i^2$ and $B^2 = \sum_i B_i^2$ stand for the Casimir operators of the Lie algebras $asu(2)$ and $bsu(2)$, of type $su(2)$, spanned by $\{A_i : i = 1, 2, 3\}$ and $\{B_i : i = 1, 2, 3\}$, respectively, where

$$A_i = \tfrac{1}{2}(L_i + N_i), \quad B_i = \tfrac{1}{2}(L_i - N_i), \quad N_i = \sqrt{\frac{-\mu}{2E}} \, M_i. \tag{29}$$

In eq. (29), L_i ($i = 1, 2, 3$) and M_i ($i = 1, 2, 3$) denote the components of the angular momentum operator and the Laplace–Runge–Lenz–Pauli vector operator, respectively.

The transition from the ordinary hydrogen atom to a q-deformed hydrogen atom can be achieved by passing from the (direct sum) Lie algebra $asu(2) \oplus bsu(2) \sim so(4)$ to the quantum algebra $asu_q(2) \oplus bsu_q(2)$. The application of this deformation to eq. (28) leads to the q-analogue of the hydrogen-like atom whose energy spectrum is given by

$$E \equiv E_j = \frac{1}{4[j][j+1]+1} \, E_0, \quad 2j \in \mathbb{N}, \tag{30}$$

where

$$E_0 = -\frac{1}{2} \frac{\mu Z^2 e^4}{\hbar^2} \tag{31}$$

is the energy of the ground state.

The q-deformed atom thus defined has the same ground-state energy as the ordinary atom. The other states are shifted when passing from $q = 1$ to $q \neq 1$. The whole (discrete) spectrum of the q-deformed hydrogen atom exhibits the same degeneracy as the ordinary one. Of course, the q-deformed spectrum coincides with the ordinary one when q goes to 1.

To close this section, we should mention there are other ways to define a q-analogue of the hydrogen atom which do not lead to the spectrum (30), (31). In this respect, by using the Kustaanheimo–Stiefel transformation (see ref. [15]), we are left with a q-deformed hydrogen atom characterized by the discrete spectrum

$$E \equiv E_{n_1 n_2 n_3 n_4} = \frac{16}{v(n_1 n_2 n_3 n_4)^2} \, E_0,$$

$$v(n_1 n_2 n_3 n_4) = \sum_{i=1}^{4} [n_i] + [n_i + 1], \quad n_i \in \mathbb{N} \quad (i = 1, 2, 3, 4). \tag{32}$$

Equation (32) can be derived (i) by transforming the three-dimensional hydrogen atom into a four-dimensional isotropic harmonic oscillator by means of the Kustaanheimo–Stiefel transformation [15], (ii) by passing from the latter oscillator to its q-analogue, and (iii) by invoking the "inverse" Kustaanheimo–Stiefel transformation. The result (32) thus obtained constitutes an alternative to (30).

5. Closing remarks

In the present paper, we have concentrated on q-deformations of three dynamical systems (harmonic oscillator, angular momentum and hydrogen atom) largely used

in physical chemistry. The q-deformed dynamical systems have been introduced in connection with the quantum algebra $su_q(2)$, which turns out to be a deformation of $su(2)$ characterized by the deformation parameter q.

We have seen that the parameter q enters the (energy) spectra for the q-analogues of the considered dynamical systems. There is no universal significance of the parameter q. However, in view of the fact that the limiting case $q = 1$ gives back the usual spectra, the deformation parameter q might be considered as a fine structure parameter (like a curvature constant), to be obtained from a fitting procedure, for describing small effects. In addition, it may happen in some situations that it is worthwhile to consider q as a completely free parameter with values far from 1, leading to new models [16].

We have experienced that the correspondence between the hydrogen atom and its q-analogue is not one-to-one. (This is indeed a general problem we face when dealing with q-analogues.) As a remedy, the use of the q-derivative leading to a q-deformed Schrödinger equation might be interesting. Also, the use of sets of noncommuting q-bosons might be appropriate to ensure $su_q(2)$ covariance.

Appendix 1

In this appendix, we give some formulas useful for dealing with q-deformed numbers $[c]$ when c are real numbers or integers.

From eq. (3), we easily obtain

$$\lim_{q \to 1} [c]_q = c;$$

$$[-c]_q = -[c]_q, \quad [c]_{1/q} = [c]_q;$$

$$[c]_q \geq c \quad \text{for} \quad c > 1.$$

Furthermore, the following relations

$$[a + b] = [a]q^b + q^{-a}[b],$$

$$[a + 1][b + 1] - [a][b] = [a + b + 1],$$

$$[a][b + c] = [a + c][b] + [a - b][c],$$

$$[a]^2 - [b]^2 = [a - b][a + b]$$

hold for any (real) numbers a, b and c.

In the case where n is a positive integer, we have

$$[n] = \sum_{i=(1-n)(2)(n-1)} q^i = q^{n-1} + q^{n-3} + \ldots + q^{-n+3} + q^{-n+1}, \quad n \in \mathbb{N} - \{0\},$$

and we can define the factorial of $[n]$ as

$$[n]! = [n][n-1]\ldots[1], \quad n \in \mathbb{N}, \quad [0]! = 1,$$

As illustrative examples, we have

$$[0] = 0, \quad [1] = 1, \quad [2] = q^{-1} + q, \quad [3] = q^{-2} + 1 + q^2, \quad [4] = q^{-3} + q^{-1} + q + q^3$$

and

$$[2][2] = [1] + [3], \quad [2][3] = [2] + [4], \quad [3][3] = [1] + [3] + [5],$$

which is reminiscent of the addition rule for angular momenta.

In the case where q is a root of unity, we have

$$q = \exp\left(i2\pi \frac{k_1}{k_2}\right), \quad k_1 \in \mathbb{N}, \quad k_2 \in \mathbb{N},$$

$$[c] = \sin\left(2\pi \frac{k_1}{k_2} c\right) \bigg/ \sin\left(2\pi \frac{k_1}{k_2}\right).$$

For instance,

$$k_1 = 1, \quad k_2 = 4 \Rightarrow q = i = \sqrt{-1} \Rightarrow [0] = [2] = [4] = \ldots = 0,$$

so that $[c] = 0$ can occur for $c \neq 0$.

Appendix 2

It is a simple matter of calculation to determine the commutation relations between the ten operators of type J, K and k defined in section 3. We list in the following only the nonvanishing commutators. The arrow indicates the limit when q goes to 1.

Nonvanishing $[k, k]$ matrix elements:

$$[k_+^+, k_-^-] = -[2K_3 + 2J_3 - 1] - [2K_3 + 2J_3 + 1] \rightarrow -4(K_3 + J_3),$$

$$[k_-^+, k_+^-] = -[2K_3 - 2J_3 - 1] - [2K_3 - 2J_3 + 1] \rightarrow -4(K_3 - J_3).$$

Nonvanishing $[J, K]$ matrix elements:

$$[J_+, K_+] = k_+^+([K_3 - J_3 - \tfrac{1}{2}] - [K_3 - J_3 + \tfrac{1}{2}]) \rightarrow -k_+^+,$$

$$[J_+, K_-] = k_+^-([K_3 + J_3 - \tfrac{1}{2}] - [K_3 + J_3 + \tfrac{1}{2}]) \rightarrow -k_+^-,$$

$$[J_-, K_+] = k_-^+([K_3 + J_3 + \tfrac{1}{2}] - [K_3 + J_3 - \tfrac{1}{2}]) \rightarrow +k_-^+,$$

$$[J_-, K_-] = k_-^-([K_3 - J_3 + \tfrac{1}{2}] - [K_3 - J_3 - \tfrac{1}{2}]) \rightarrow +k_-^-.$$

Nonvanishing $[J, k]$ matrix elements:

$$[J_3, k_+^+] = k_+^+ \quad [J_3, k_-^+] = - k_-^+ \quad [J_3, k_-^-] = - k_-^- \quad [J_3, k_+^-] = k_+^-,$$

$$[J_+, k_-^+] = K_+([K_3 - J_3 + \tfrac{3}{2}] - [K_3 - J_3 - \tfrac{1}{2}]) \rightarrow + 2K_+,$$

$$[J_+, k_-^-] = K_-([K_3 + J_3 + \tfrac{1}{2}] - [K_3 + J_3 - \tfrac{3}{2}]) \rightarrow + 2K_-,$$

$$[J_-, k_+^+] = K_+([K_3 + J_3 - \tfrac{1}{2}] - [K_3 + J_3 + \tfrac{3}{2}]) \rightarrow - 2K_+,$$

$$[J_-, k_+^-] = K_-([K_3 - J_3 - \tfrac{3}{2}] - [K_3 - J_3 + \tfrac{1}{2}]) \rightarrow - 2K_-.$$

Nonvanishing $[K, k]$ matrix elements:

$$[K_3, k_+^+] = k_+^+ \quad [K_3, k_-^+] = k_-^+ \quad [K_3, k_-^-] = - k_-^- \quad [K_3, k_+^-] = - k_+^-,$$

$$[K_+, k_-^-] = J_-([K_3 + J_3 + \tfrac{1}{2}] - [K_3 + J_3 - \tfrac{3}{2}]) \rightarrow + 2J_-,$$

$$[K_+, k_+^-] = J_+([K_3 - J_3 - \tfrac{3}{2}] - [K_3 - J_3 + \tfrac{1}{2}]) \rightarrow - 2J_+,$$

$$[K_-, k_+^+] = J_+([K_3 + J_3 - \tfrac{1}{2}] - [K_3 + J_3 + \tfrac{3}{2}]) \rightarrow - 2J_+,$$

$$[K_-, k_-^+] = J_-([K_3 - J_3 + \tfrac{3}{2}] - [K_3 - J_3 - \tfrac{1}{2}]) \rightarrow + 2J_-.$$

From the commutation relations in this appendix and in section 3, we recover that the set $\{J, K, k\}$ spans the (10-dimensional) noncompact Lie algebra $so(3, 2) \sim sp(4, \mathbb{R})$ in the limiting case $q = 1$ [12].

Acknowledgements

References

[1]   V.G. Drinfel'd, Sov. Math. Dokl. 32(1985)254.
[2]   M. Jimbo, Lett. Math. Phys. 10(1985)63.
[3]   S.L. Woronowicz, Commun. Math. Phys. 111(1987)613.
[4]   A.J. Macfarlane, J. Phys. A22(1989)4581.
[5]   L.C. Biedenharn, J. Phys. A22(1989)L873.
[6]   C.-P. Sun and H.-C. Fu, J. Phys. A22(1989)L983.
[7]   P.P. Kulish and N.Yu. Reshetikhin, Lett. Math. Phys. 18(1989)143.
[8]   M. Nomura, J. Phys. Soc. Japan 59(1990)2345.
[9]   C. Quesne, Phys. Lett. A153(1991)303.

[10] J. Katriel and A.I. Solomon, J. Phys. A24(1991)2093.

[11] J. Schwinger, On angular momentum, Report US AEC NYO-3071, 1952, in: *Quantum Theory of Angular Momentum*, ed. L.C. Biedenharn and H. van Dam (Academic Press, New York, 1965).

[12] M. Kibler and G. Grenet, J. Math. Phys. 21(1980)422.

[13] Yu.I. Kharitonov, Yu.F. Smirnov and V.N. Tolstoy, Method of the projection operators and q-analog of the quantum angular momentum theory, Reports 1607, 1636, Institute of Nuclear Physics, Academy of Sciences of the USSR, Leningrad (1990).

[14] W. Pauli, Z. Phys. 36(1926)336.

[15] M. Kibler and T. Négadi, Lett. Nuovo Cimento 37(1983)225; J. Phys. A16(1983)4265; Phys. Rev. A29(1984)2891.

[16] J.A. Tuszyński and M. Kibler, work in progress.

[17] Y. Saint-Aubin, Quantum groups and their application to conformal quantum field theories, Report CRM-1663, Université de Montréal (1990).